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The Leibniz Principle in Quantum Logic

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The principle of the "identity of indiscernibles" (Leibniz Principle) is investigated within the framework of the formal language of quantum physics, which is given by an orthomodular lattice. We show that the validity of this principle is based on very strong preconditions (concerning the existence of convenient predicates) which are given in the language of classical physics but which cannot be fulfilled in orthomodular quantum logic.

1. INTRODUCTION

The "principium identitatis indiscernibilium" was first formulated by Leibniz as a metaphysical principle. This principle—referred to here as "*Leibniz principle*"—states that two objects which cannot be distinguished by any interior property are at all identical (Leibniz, 1875–1890, Vol. V, p. 401, Vol. VI, p. 608). Leibniz illustrated the plausibility of the "identity of indiscernibles" by several examples, partly from everyday life and partly from mathematics. In his writings there are at least two proof attempts for the principle, both of which are not really convincing. (1) A formal proof: If one assumes that the set of properties which pertain to an object contains a naming predicate which is unique with respect to the object, then the indistinguishability implies the identity (Lorenz, 1969). (2) A theological proof: In the actual world there are no two objects with all properties in common, since for the creator of the world there was no sufficient reason to create the same thing twice (Leibniz, 1875–1890, Vol. VI, p. 371).

The present paper will not be concerned with the question of whether the properties of a class of real objects are such that the Leibniz principle holds for this class. This may happen for objects which belong to the domain of classical physics, but presumably it does not happen for the class of quantum mechanical objects (Mittelstaedt, 1985, 1986). Here we investigate the formal language $\mathscr{L}_0$ of quantum physics, sometimes called quantum logic, with respect to the question of whether in $\mathscr{L}_0$ the Leibniz principle

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holds as a theorem. This problem is treated here exclusively by logical and semantical means, making use of the well-known properties of $\mathscr{L}_0$, but without any recourse to quantum mechanics as a physical theory and without making use of experimental facts.

Along this line of investigation, there are some recent papers by Dalla Chiara (1985, 1986) and Dalla Chiara and Toraldo di Francia (1985) in which the validity of the Leibniz principle for quantum objects is treated from a physical aspect (Dalla Chiara and Toraldo di Francia, 1985) as well as from a semantical point of view (Dalla Chiara, 1985, 1986). It turns out that an algebraic or Kripkean semantics for second-order quantum logic does not automatically provide a violation of the Leibniz principle. For fermions a quaset-theoretical semantics for a second-order language can be constructed in which the Leibniz principle holds, whereas in the case of bosons we are faced with a violation of the Leibniz principle. The invalidity of this principle can be further illustrated by comparing the semantical models of bosons with Henkin-style semantics for second-order classical logic (Scott, 1969) in which the Leibniz principle fails. The metalogical reason why the Leibniz principle fails is the same in both cases, namely the restriction to a proper subset of all properties of a model. But whereas such a restriction is quite artificial in second-order classical logic, it seems to be strongly suggested in the case of models for bosons by the physical properties of such particles.

In contrast to these investigations, which treat the validity of the Leibniz principle from a semantical point of view, making use of some contingent properties of quantum systems, e.g., the distinction between Bose and Fermi particles, in the present paper the Leibniz principle is investigated under a purely syntactical aspect. Quantum logic as a second-order object language is taken here for granted. We then show that the Leibniz principle can be proved in classical logic as well as in quantum logic under the premise that for any object there exists a naming predicate which is unique with respect to the other object. If within the framework of the formal language objects are introduced by definite descriptions, the premise mentioned holds in classical logic but it cannot be fulfilled in quantum logic. Hence we arrive at the conclusion that the Leibniz principle holds in classical logic for objects constituted by definite descriptions, but that it is not valid in quantum logic.

2. THE LEIBNIZ PRINCIPLE IN FORMAL LANGUAGES

2.1. The Formal Language

We consider an *orthomodular lattice* $\mathscr{L}_0$ of proposition $A \in \mathscr{L}_0$ which state that a predicate P holds for an object x, i.e., $A = P(x)$. Objects

$x, y, \ldots$ correspond to quantum systems and predicates $P, Q, \ldots$ to observable properties, such as spin, position, angular momentum etc. The lattice $\mathscr{L}_0$ will often be assumed to fulfill the additional law $a(\mathscr{L}_0)$ of atomicity:

> For any proposition $A \in \mathscr{L}_0$ with $A \neq 0$ there exists an atom $W \in \mathscr{A}(\mathscr{L}_0)$, where $\mathscr{A}(\mathscr{L}_0)$ is the set of atoms of $\mathscr{L}_0$, such that $W \leq A$.

and the covering law $c(\mathscr{L}_0)$:

> If $A \in \mathscr{L}_0$ and $W \in \mathscr{A}(\mathscr{L}_0)$ satisfy $W \wedge A = 0$, then for any $X \in \mathscr{L}_0$ with $A \leq X \leq A \vee W$ it follows that $X = A$ or $X = A \vee W$.

The properties $a(\mathscr{L}_0)$ and $c(\mathscr{L}_0)$ are necessary conditions which must be fulfilled if all propositions $A \in \mathscr{L}_0$ can be related to a single object S as a referent such that $A = P(S)$ (Stachow, 1985). Hence we assume here mostly that the conditions $a(\mathscr{L}_0)$ and $c(\mathscr{L}_0)$ hold for $\mathscr{L}_0$. However, it turns out that even the strong requirements $a(\mathscr{L}_0)$ and $c(\mathscr{L}_0)$ are not sufficient for the validity of the Leibniz principle. Moreover, it will be shown that for an atomic lattice $\mathscr{L}_0$ which fulfills the covering law, the most important premise of the Leibniz principle can be refuted.

2.2. The Leibniz Principle

Let $\mathscr{S}$ be a set of objects $x, y, \ldots$, $\mathscr{P}$ a set of predicates $P, Q, \ldots$; and $\mathscr{L}$ a lattice of proposition $A = P(x)$. The "identity" of objects will be denoted by $=$. Then the *Leibniz principle* reads[2]

Theorem 1. (LP) For any elements $x, y \in \mathscr{S}$: If $x \neq y$ is true, then there exists a predicate P, such that $1 \leq P(x)$ and $1 \leq \neg P(y)$.

Proof. Assume (LP) does not hold. Then there exist elements x, y such that $x \neq y$ is true and for any predicate $P \in \mathscr{P}$ the implication $P(x) \leq P(y)$ holds. Let (x, y) be a pair of elements for which this is the case and $P \in \mathscr{P}$ be the predicate "being an x," i.e., $P(z) :=z \in \{x\}$. Then we get $1 \leq x \neq y$ and $P(x) \leq P(y)$, and because of $1 \leq P(x)$ it follows $1 \leq P(y)$ and thus $x \neq y \wedge P(y) \leq 0$, since $y \in \{x\}$ is false for $x \neq y$. ■

[2]Here we use the following terminology. The equivalence relation is denoted by $=$ and the implication relation by $\leq$. For the "true proposition" we write 1 and for the "false proposition" 0. Hence the truth of a proposition A can be expressed by $1 \leq A$ and its falsity by $A \leq 0$.

This proof is based on the premise that for any $x \in \mathcal{S}$ there exists a predicate P (to be an x) such that $P(x)$ is true and $x \neq y \wedge P(y) \leq 0$. If we formulate this premise (PLP) of the Leibniz principle explicitly, it reads

> PLP: For any object $x \in \mathcal{S}$ there exists a predicate P, such that $P(x)$ is true and for any $y \in \mathcal{S}$ it holds $x \neq y \wedge P(y) \leq 0$.

It is easy to show that we have the following result.

Theorem 2. (WLP) The premise (PLP) implies the Leibniz principle (LP). This theorem will be called the "*weak Leibniz principle*" (WLP).

In the next section we first discuss the validity of the "weak Leibniz principle" (WLP) in $\mathcal{L}_B$ and in $\mathcal{L}_0$. Its meaning will be discussed later.

2.3. Discussion of the WLP

The WLP states that under the premise (PLP) the Leibniz principle LP holds. The proof makes use of the rule (in propositional logic)

$$x \neq y \wedge P(y) \leq 0 \Rightarrow x \neq y \leq \neg P(y) \tag{I}$$

which is true in $\mathcal{L}_B$ but not generally in $\mathcal{L}_0$. Here however we are dealing with the special propositions $x \neq y$, $P(y)$, and $\neg P(y)$ which have the following properties:

$x \neq y \sim P(y)$ and $x \neq y \sim \neg P(y)$, i.e., $x \neq y$ and $P(y)$ are commensurable,[3] where P is the predicate "to be an x." If $P(y)$ and $x \neq y$ were not commensurable, one could not decide whether the object x with $1 \leq P(x)$ is different from y, i.e., whether $x \neq y$ is true.

For these reasons we have the commensurability relations:

$$x \neq y \sim P(y), \qquad x \neq y \sim \neg P(y)$$

In orthomolecular logic the rule $A \wedge B \leq 0 \Rightarrow A \leq \neg B$ can be proved if the two commensurability relations $A \sim B$ and $A \sim \neg B$ are presupposed (Mittelstaedt, 1978). Hence, the relation (I) is valid even in the orthomodular

[3]Here we use the notation $A \sim B$ for the (symmetric) commensurability relation $A = (A \wedge B) \vee (A \wedge \neg B)$ (Mittelstaedt, 1978, pp. 31ff).

logic $\mathscr{L}_0$ and thus the *weak Leibniz principle* holds not only in $\mathscr{L}_B$ but also in the weaker logic $\mathscr{L}_0$.

3. THE NAMING PREDICATE

In the languages $\mathscr{L}_B$ and $\mathscr{L}_0$ the Leibniz principle (LP) is valid if for any $x \in \mathscr{S}$ there exists a predicate N which pertains to x and which is unique with respect to the object (PLP). This predicate N will be called the *naming predicate* of x. In this section we investigate some properties of naming predicates.

In a physical theory like quantum mechanics, objects are not given as primitive elements but must be constituted by means of "essential" and "accidential" properties (Mittelstaedt, 1985, 1986). In a similar way in quantum logic $\mathscr{L}_0$ objects $x \in \mathscr{S}$ must be introduced by means of definite descriptions. In the present situation this can be done with naming predicates, i.e., an object S is given by

$$S = (\iota x) N_S(x)$$

where N_S is the naming predicate "to be an S." For the naming predicate N_S we have the uniqueness property[4]

$$1 \leq N_S(S) \wedge \bigvee_{S'} (N_S(S') \to S = S') \tag{UN}$$

and for any other predicate $B \in \mathscr{P}(\mathscr{L})$ Russell's formula holds,

$$B(S) = \underset{x}{\exists} \left\{ N_S(x) \wedge B(x) \wedge \underset{y}{\forall} N_S(y) \to x = y \right\}$$

Within the framework of the language $\mathscr{L}_0$, a naming predicate must fulfill two additional requirements, as follows.

1. *The name N is an atom of* $\mathscr{L}_0$. If $S = (\iota x) N_S(x)$ and N_S is *not* an atom, then there exists an atom A s.t. $0 < A < N_S$. Hence for the element $a = (\iota x) A(x)$ we have also $N_S(a)$ and thus according to (UN), $S = a$. Furthermore, from $S = a$ and $A(a)$ we get $A(S)$ and due to the uniqueness $1 \leq \forall_x (N_S(x) \to A(x))$ and thus $N_S \leq A$, which contradicts the assumption.

2. *The name N is in the center* $\mathscr{Z}(\mathscr{L}_0)$. If N is an atom, $N \in \mathscr{A}(\mathscr{L})$, and if $\mathscr{L} = \mathscr{L}_B$ is a Boolean lattice, then for any $A \in \mathscr{L}_B$ one has

$$N \leq A \qquad \text{or} \qquad N \leq \neg A \tag{D}$$

[4]If in a physical theory the position predicate, say, satisfies this requirement, the uniqueness property is called "impenetrability" (Mittelstaedt, 1985, 1986).

which means that an object $n=(\imath x)N(x)$ possesses any other property $A \neq N$ or its negation. In orthomodular logic $\mathcal{L}_0$ this determination property (D) of a name is only given if $N \sim A$; i.e., if the name N and the predicate A are commensurable.

Thus we have the following result.

Theorem 3. For any proposition $A \in \mathcal{L}_0$ and any atom $N \in \mathcal{A}(\mathcal{L}_0)$ s.t. $A \sim N$, *it follows that*

$$N \leq A \qquad \text{or} \qquad N \leq \neg A$$

Consequently, a name $N \in \mathcal{A}(\mathcal{L}_0)$ which fulfills the condition (D) for any $A \in \mathcal{L}_0$ must be commensurable with any A and thus belongs to the center $\mathcal{Z}(\mathcal{L}_0)$ of $\mathcal{L}_0$.

It is worthwhile to note that a Boolean lattice $\mathcal{L}_B^{(k)}$, which is generated by a finite number of elements $G_1, G_2, \ldots, G_k$ is atomic and the elements $N^{(\nu)} \in \mathcal{A}(\mathcal{L}_B^{(k)})$ are given by $N^{(\nu)} = G_1^{(\nu)} \wedge G^{(\nu)} \wedge \cdots \wedge G_k^{(\nu)}$ with $G_i^{(\nu)} \in \{G_i, \neg G_i\}$. The atoms $N^{(\nu)} \in \mathcal{A}(\mathcal{L}_B^{(k)})$ then satisfy the condition (D) and since $\mathcal{Z}(\mathcal{L}_B) = \mathcal{L}_B$, all atoms are in the center of $\mathcal{L}_B^{(k)}$. Hence the requirements of the premise (PLP) are fulfilled and we arrive at the following interesting result: If within the framework of a Boolean language $\mathcal{L}_B^{(k)}$ objects S_i are given by definite descriptions $S_i = (\imath x)N_i(x)$, where N_i are naming predicates, then the Leibniz principle holds for these objects as a theorem.[5]

4. NAMING IN ORTHOMODULAR LOGIC

According to the previous discussion, a naming predicate N is an atom, $N \in \mathcal{A}(\mathcal{L}_0)$, and is in the center $N \in \mathcal{Z}(\mathcal{L}_0)$. In this section we prove that there exists no naming predicate in $\mathcal{L}_0$ with these properties.

We will distinguish two cases: (i) an orthomodular lattice *does not admit superselection rules,* (ii) an orthomodular lattice *admits superselection rules.*

Definition. An orthomodular lattice $\mathcal{L}_0$ does *not* admit superselection rules iff for any pair of atoms W_1, W_2 of $\mathcal{L}_0$ ($W_1 \neq W_2$), there exists an atom W_3 of $\mathcal{L}_0$ s.t. (i) $W_1 \neq W_2$, $W_1 \neq W_3$, and $W_2 \neq W_3$; (ii) $W_1 \vee W_2 = W_1 \vee W_3 = W_2 \vee W_3$.

Lemma. Let $\mathcal{L}_0$ be an atomic orthomodular lattice. If $\mathcal{L}_0$ does not admit superselection rules, then $\mathcal{L}_0$ is not a Boolean lattice.

[5]The way of reasoning which leads to this surprising result reminds one of the "transcendental arguments" In Kant's critique of pure reason. Hence one could consider the Leibniz principle as a "synthetic judgement *a priori*" which holds for individual objects constituted by definite descriptions.

Proof. Let W_1, W_2, W_3 be three atoms as in the definition. Let us suppose, by contradiction, that $\mathscr{L}_0$ is a Boolean lattice. Then $W_3 \wedge (W_1 \vee W_2) = (W_3 \wedge W_1) \vee (W_3 \wedge W_2)$; but $W_3 \wedge W_1 = 0 = W_3 \wedge W_2$, since W_1, W_2, W_3 are atoms. Now, $W_3 \leq W_1 \vee W_2$, since $W_1 \vee W_2 = W_1 \vee W_3$ and therefore $W_3 \wedge (W_1 \wedge W_2) = W_3$; Hence, $W_3 = 0$, contradiction $\|$ ■

Definition. An orthomodular lattice $\mathscr{L}_0$ is called irreducible if the center $\mathscr{Z}(\mathscr{L}_0)$ is trivial. In all other cases the lattice is called reducible.

Theorem 4. Let $\mathscr{L}_0$ be an atomic orthomodular complete lattice with the covering property. $\mathscr{L}_0$ does not admit superselection rules *iff* it is irreducible.

Proof. See Beltrametti and Cassinelli (1981) or Kalmbach (1983, p. 142, Theorem 8). ■

Theorem 5. Let $\mathscr{L}_0$ be an atomic complete orthomodular lattice with the covering property not admitting superselection rules. Then there exists no naming predicate N s.t. N is an atom of $\mathscr{L}_0$ and $N \in \mathscr{Z}(\mathscr{L}_0)$.

Proof. Theorem 5 is a direct consequence of Theorem 4. If an atomic orthomodular lattice $\mathscr{L}_0$ with the covering property *admits* superselection rules, then it is no longer irreducible. However, it can be represented as direct sum of irreducible orthomodular lattices. In particular, $\mathscr{L}_0$ is the direct sum of the segments $\mathscr{L}_0[0, z_i]$, where z_i are atoms of the center of $\mathscr{L}_0$. ■

We prove now the existence of orthomodular lattices with superselection rules where it is not possible to define any naming predicate belonging to the center.

Let $\{\mathscr{H}_i\}$ be a sequence of Hilbert spaces s.t. $\dim(\mathscr{H}_i) \geq 2$, for any i and $\mathscr{H} := \bigoplus \mathscr{H}_i$. It is well known that $\mathscr{H}$ is still a Hilbert space. Let $\mathscr{P}(\mathscr{H})$ be the orthomodular lattice of all projections of $\mathscr{H}$ and

$$\mathscr{P}(\mathscr{H})\# := \left\{ P \in \mathscr{P}(\mathscr{H}) / P = \bigvee_i P \wedge P_i, \text{ where } P_i = P_{\mathscr{H}_i} \right\}$$

Then it is not hard to prove that $P(\mathscr{H})\#$ *is an orthomodular lattice with superselection rules.* Furthermore, $\mathscr{P}(\mathscr{H})\#$ is the direct sum of the segments $\mathscr{L}[0, \mathscr{P}(\mathscr{H}_i)]$ which are, of course, irreducible. Then (Beltrametti and Cassinelli, pp. 101, 133) every $\mathscr{P}(\mathscr{H}_i)$ is an atom of the center of $\mathscr{P}(\mathscr{H})\#$. Now, the atoms of $\mathscr{Z}(\mathscr{P}(\mathscr{H})\#$ are the central covers of the atoms of $\mathscr{P}(\mathscr{H})\#$. But, then, there exists no atom of $P(\mathscr{H})\#$ belonging to the center of $\mathscr{P}(\mathscr{H})\#$, since otherwise there would exist a direct summand of $\mathscr{P}(\mathscr{H})\#$ of length strictly less than 2.

It is possible also to show some examples of orthomodular lattices admitting naming predicates (see, for instance, the so-called orthomodular lattice[6] G_{12}). But, as we will see, even if a set of naming predicates exists, it cannot be *sufficient*, where a set $\mathcal{N}$ of naming predicates is said to be sufficient *iff* for any $A \neq 0$ ($A \in \mathcal{L}_0$) there is a $W \in \mathcal{N}$ s.t. $W \leq A$. This is the content of the following results.

Theorem 6. Let $\mathcal{L}_0$ be a complete atomic orthomodular lattice with the covering property admitting superselection rules. Then there exists no sufficient set of naming predicates.

Proof. Let us suppose, by contradiction, the existence of a sufficient set $\mathcal{N}$ of naming predicates in $\mathcal{L}_0$. Then $\forall A \neq 0$, $\exists W \in \mathcal{N}$ s.t. $W \leq A$. But W is an atom of $\mathcal{L}_0$ belonging to $\mathcal{Z}(\mathcal{L}_0)$. Therefore, $\forall A \in \mathcal{L}_0$: $W \leq A$ or $W \leq \neg A$. Thus, the filter $F_W =: \{B \in \mathcal{L}_0 / W \leq B\}$ is a proper and complete filter of $\mathcal{L}_0$ and, accordingly, the map $h: \mathcal{L}_0 \to \mathcal{B}_2$ (where $\mathcal{B}_2$ is the two-element Boolean algebra) defined as

$$h(A) = \begin{cases} 1, & \text{if } A \in F_W \\ 0 & \text{otherwise} \end{cases}$$

is a homomorphism from $\mathcal{L}_0$ into $\mathcal{B}_2$. Therefore, every naming predicate in N determines a homorphism from L_0 into $\mathcal{B}_2$. Let $\mathcal{N}\#$ be the set of all such homomorphisms. It is easy to see that $\mathcal{N}\#$ is sufficient, i.e., $\forall A \neq 0$,

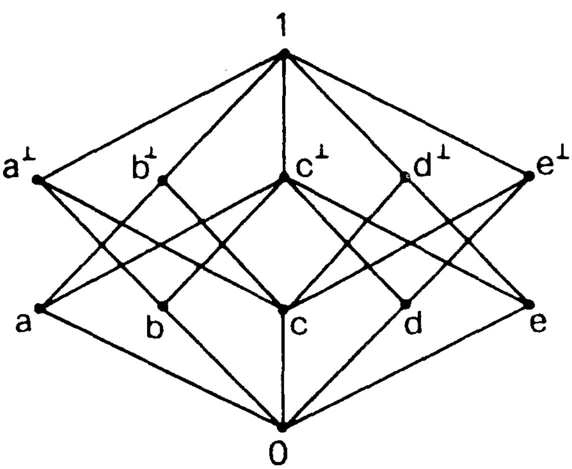

Fig. 1. The orthomodular lattice G_{12}.

[6]The center of G_{12} is $\{0, 1, c, c^{\perp}\}$. Therefore, c is an atom in the center of G_{12} (Fig. 1).

$\exists h \in \mathcal{N}\#$ s.t. $h(A)=1$. If $A \neq 0$, indeed $\exists W \in \mathcal{N}$ s.t. $W \leq A$; but $h_W(W)=1$ (h_W is the homomorphism determined by W). Hence $h_W(A)=1$, since $W \leq A$. Let Z be an atom of the center of $\mathcal{L}_0$. Then, $\exists h \in \mathcal{N}\#$ s.t. $h(Z)=1$. But then h is a homomorphism from the segment $\mathcal{L}[0, Z]$ into $\mathcal{B}_2$, since, for any $A, B \in \mathcal{L}[0, Z]$, $h(A \wedge B) = h(A) \cap h(B)$ and $h(A\#) = h(\neg A \wedge Z) = h(\neg A) = -h(A)$, where $\#$ is the relative complementation with respect to which $\mathcal{L}[0, Z]$ is an ortholattice. Now, it is not hard to prove that the element $A^* = \bigwedge A_i$, where $A_i \in \{A \in \mathcal{L}[0, Z] / h(A) = 1\}$ is an atom of $\mathcal{L}[0, Z]$ which belongs to the center of $\mathcal{L}[0, Z]$. But $\mathcal{L}[0, Z]$ is an atomic, complete, and irreducible orthomodular lattice with the covering property and therefore, by Theorem 5, no atom can exist in its center, ↯. It is also interesting to see that the existence of a naming predicate in the center of an orthomodular lattice $\mathcal{L}_0$ implies the nonrefutability of any tautology of the classical propositional calculus. This is the content of the following results.

Theorem 7. Let $\mathcal{L}_0$ be an atomic orthomodular lattice with the covering property and admitting superselection rules. If there exists a naming predicate in the center of $\mathcal{L}_0$, then every tautology of the classical propositional calculus is not refuted in $\mathcal{L}_0$.

Proof. It is a slight generalization of the proof contained in Giuntini (1987). ■

5. CONCLUDING REMARKS

Within the framework of a formal language, the Leibniz principle connects the identity of objects with the indiscernibility by means of predicates. Without further assumptions, nothing can be said in general about the validity of this principle. Here we made use of the fact that object systems in physics are constituted by means of properties, which means in the formal language that objects are given by definite descriptions. Under this premise the Leibniz principle holds in the language of classical physics but it is no longer valid in orthomodular quantum logic. The assumption that the premise of the Leibniz principle were fulfilled in quantum logic could be formally disproved. Hence our conclusion is that the Leibniz principle cannot be proved in the usual way.

REFERENCES

Beltrametti, E., and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Dalla Chiara, M. L. (1985). Some foundational problems in mathematics suggested by physics *Synthese*, **62**, 303-315.

Dalla Chiara, M. L. (1986). Quantum logic, in *Handbook of Philosophical Logic* D. Gabbay and F. Guenthner, eds., Reidel, Dordrecht, Vol. III, pp. 427–469.

Dalla Chiara, M. L., and Toraldo di Francia, G. (1985). Individuals, kinds and names in physics, *Versus*, **40**, 29–50.

Giuntini, R. (1987). Quantum logics and Lindenbaum property, *Studia Logica*, XLVI (1), 17–35.

Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, London.

Leibniz, G. W. (1875–1890). *Philosophische Schriften*, C. I. Gerhard, ed., Berlin.

Lorenz, K. (1969). Die Begründung des principium identitatis indiscernibilium, in *Akten des internationalen Leibniz-Kongresses*, Vol. III, *Hannover 1966*, Franz Steiner Verlag, Wiesbaden.

Mittelstaedt, P. (1985). Constituting, naming and identity in quantum logic, in *Recent Developments in Quantum Logic*, P. Mittelstaedt and E. W. Stachow, eds., Bibliographisches Institute, Mannheim.

Mittelstaedt, P. (1986). Naming and identity in quantum logic, in *Foundations of Physics*, P. Weingartner and G. Dorn, eds., Hölder-Pichler-Tempski, Vienna.

Mittelstaedt, P. (1978). *Quantum Logic*, Reidel, Dordrecht.

Scott, D. (1969). *Advice on modal logic, in Philosophical Problems in Logic* K. Lambert, ed., Reidel, Dordrecht.

Stachow, E. W. (1985). Structures of quantum language for individual systems, in *Recent Developments in Quantum Logic*, P. Mittelstaedt and E. W. Stachow, eds., Bibliographisches Institut, Mannheim.